and anticyclones. Thus, one can see by referring to my charts 14–25 in the MONTHLY WEATHER REVIEW of February, 1905, that the mean daily gradients can be found only by taking the daily means at the several levels and computing the gradients from them. Gradients referred to the summit of Blue Hill, 195 meters, will be very different from those referred to the Valley Station, 15 meters, and the gradients will differ greatly for the same day, according to the hour to which the observations refer. Much of the divergence in the published gradient data from the several countries is due to an omission to apply these principles. In the case of the Blue Hill data, the record for the Valley Station was usually published only for the beginning and the ending of a kite flight, and it was not possible to accurately interpolate the intermediate values from the report. These temperature data were courteously supplied by the observatory to the Weather Bureau by a special arrangement.

Applying these considerations to the Mount Weather Observatory, it is evident that the summit station must be supplemented by self-registering instruments at the low levels, in the Piedmont country at Trapp and Leesburg and in the Shenandoah Valley at Berryville, Winchester, and Front Royal. There is certain to be a dynamic heating and cooling effect as the wind sweeps over the summit of the Blue Ridge Mountains, and the local action of the currents must be carefully investigated. Furthermore, there is much danger of failing to deduce the correct values of the gradients from the balloon ascensions unless the temperatures at the level of the country beneath them from Bluemont to Washington and Baltimore can be secured, and unless the balloons are sent up in about equal numbers at all hours of the day and night. It is necessary to eliminate all the local conditions and the ordinary hourly variations before anything like accurate general temperature gradients can be computed.

(3) *The height of the diurnal convectional disturbances.*—Mr. Clayton seems to have assigned 2000 meters as the height at which the diurnal variation of the temperature becomes insensible, but by reference to Table XIII of his report, it is seen that the gradients in the successive 500-meter levels do not become constant for each month, as between the day and night values given in the first and second sections of the table, till the 2500–3000 stratum is passed. It has been my experience that the vertical gradients on the New England coast are generally smaller than they are in the middle valleys of the continent, and I am of the opinion that the diurnal convection reaches about the two-mile limit in many regions of the United States. Consequently, I have adopted that elevation as the limit in order to secure a system of gradients somewhat more applicable to the entire country than those strictly limited to the Blue Hill district could be supposed to represent.

(4) *The inversion of temperatures.*—One of the valuable facts brought out in these discussions is concerned with the diurnal inversion of temperatures in the lower strata. Except in midsummer when the air is warmed up to great heights, and the ground does not cool so rapidly during the night, there is a complete inversion between the surface temperature and the air temperature at a few hundred meters above the ground, throughout the day and night, except at the hours of transition, 8 a. m. and 8 p. m., approximately. These air conditions are the results of two movements, (1) the vertical convection over a station, and (2) the horizontal movement of the cone of temperature-fall, which is very rapid from east to west with the earth's rotation. Generally the rising warm air of the day falls back to the ground outside this cone, that is during the night, and this makes a complete inversion between the day and night throughout the 24 hours. The cold temperature by day, in the air above the warm ground, consists really of the cold night air of the preceding day, and the warm air by night over the cold surface is the warm air of the preceding day lagging behind in its ascensional and its descensional path. The semidiurnal temperature curve, however, exhibits the influence of the surface heating and cooling when the convectional currents become vigorous through short distances in elevation. These remarks may be regarded as supplementary to those contained in the Blue Hill discussion of the observations.

It will be seen that these Blue Hill kite observations are unusually fruitful and valuable in many meteorological studies, and we sincerely hope that the series may be extended to several more years, and that other observatories will add further important data of a similar kind.

MATHEMATICAL THEORY OF THE NOCTURNAL COOLING OF THE ATMOSPHERE.

By S. TETSU TAMURA.

I. HISTORICAL AND CRITICAL SURVEY OF THE PROBLEM OF THE NOCTURNAL COOLING OF THE ATMOSPHERE.

The problem of the cooling of the atmosphere belongs to the group of the most important, yet extremely difficult, problems in meteorology. The variation of atmospheric temperature, for instance, depends not solely upon the solar radiation, which varies with the sun's altitude and the degree of sunshine or cloudiness, but, as shown by Fourier, Poisson, Pouillet, Melloni, and others, it also depends upon the influence of the emission by the earth's surface, that is, by the soil, vegetation, snow, and ocean. Even as to the air itself, we have to distinguish the radiating power of the dry air, clouds, haze, and dust.

If the lower strata of the air are heated by the solar rays and the emission by the earth's surface, the heated air tends to rise. This results very soon in tremulous and flickering streams, which collect into larger ones. Thus the propagation of heat goes on with relative rapidity by the process of convection. The conduction of heat from the earth's surface to the atmosphere also exists, though it operates very slowly. At the same time, a portion of the heat thus communicated is lost by a process of atmospheric radiation into sidereal space. Again, the influence of the winds and rain upon the atmospheric temperature is by no means negligible. Thus, it appears that it is impossible for us, at least in the present state of our knowledge, to solve the problem by the light of modern mathematical analysis; for even the world's greatest mathematicians of the last century failed to solve the problem analytically and left to us the so-called Bessel's interpolation[1] formula expressed by Fourier's series, which has been criticized by Wild and Angot as too far from representing nature.

If, however, we have to deal with the problem of the nocturnal cooling of the atmosphere, or of the cooling of the calm atmosphere during long arctic nights, our problem becomes a great deal simpler. At night the earth's surface is cooled by its emission of heat into space, and the temperatures of the lower layers of the air fall, and they thus become heavier. Therefore convective movements cease to a great extent. Other disturbances also are comparatively small during the nighttime. Consequently we may consider the nocturnal cooling of the atmosphere as dependent only upon radiation and conduction of heat between the atmospheric strata and the earth's surface.

One of the first who observed the cooling of bodies exposed during the night in the open field, under a clear sky and calm atmosphere, was Patrick Wilson, of Glasgow. His observations[2] were made about 1783 by means of two thermometers, one placed on the snow, the other freely suspended at the

[1]This formula is

$$\theta = \theta_m + A_1 \sin\left(\frac{2\pi}{T}t + \phi_1\right) + A_2 \sin\left(\frac{2\pi}{T}2t + \phi_2\right) + \ldots\ldots$$

where T = periodic time, expressed in terms of hours.
θ_m = mean temperature of the day.
t = time reckoned in hours.
A_1, A_2, A_3, ϕ_1, ϕ_2, ϕ, etc., are constants.

[2]Edinburgh Phil. Trans. Vol. I, p. 153.

height of four feet. On one of these nights the lower thermometer, under a perfectly clear sky, marked —21.7° F., while the other marked —15.0° F. The difference of six degrees diminished rapidly when clouds appeared on the horizon and entirely vanished when the sky was completely covered; the two thermometers had then the same reading, —13.7° F. Wilson was also the first to show that the effect of the radiation of bodies toward the sky is sensibly the same at all temperatures; so that in nights equally calm and serene the same substance is always cooled to the same extent, whatever may be the temperature of the atmosphere.

Some years later Six[3] found that a thermometer placed on the grass of a meadow during calm and clear nights continued several degrees lower than another perfectly similar thermometer suspended at the height of five or six feet, the difference between the two amounting sometimes to 7.5° F.

At the beginning of the nineteenth century, Wells[4] performed a long series of experiments on the nocturnal cooling of bodies, by placing thermometers in contact with the ground and leaves of plants, or by enveloping them with wool, cotton, and other substances. These thermometers, placed at a small distance from the earth's surface, gave a fall of 4.5° F. and even 7.8° F. below a thermometer without any envelope suspended at the height of four feet.

The experiments of Wells were repeated by many observers and notably by M. Pouillet[5]. He inserted one of the thermometers into swans-down contained in a vessel placed on the ground and left the other thermometer at the height of four feet. The lower thermometer, on certain nights, fell 8° or 9° F. below the upper one.

In 1847 M. Melloni read to the Royal Academy of Sciences at Naples a memoir[6] on the nocturnal cooling of bodies exposed to a free atmosphere in calm and serene weather and on the resulting phenomena near the earth's surface. This interesting paper tells us that Melloni, on some September nights of 1846, in the valley named La Lava, situated between the cities of Naples and Salerno, attempted to compare the radiation of lampblack, metals, and other substances, such as sawdust, leaves, and vegetable earths, exposed to the nocturnal influence of a serene sky. He came to the conclusion that the cooling of a black thermometer is owing to radiation, and not to the contact of the external air, and that the radiation of a metallic thermometer is so feeble as to escape direct observation, and also that the emissivity of the earth and vegetable substances did not seem to differ much from that of lampblack. He also convinced himself of the fact observed by Wilson, that a body exposed during the night to the influence of a sky of equal clearness and calmness is always cooled to the same extent, whatever may be the temperature of the air.

All these investigations on the nocturnal cooling of bodies, which are directly related to the problem of nocturnal cooling of the atmosphere itself, have, however, been purely observational. For the first formula of nocturnal cooling we are indebted to Johann Heinrich Lambert, of the Academy of Berlin. In the latter part of the eighteenth century he published his celebrated work "Pyrometrie," and in it he has given the following formula:[7]

$$\theta = \theta_i + Ab^t \tag{1}$$

in which θ_i is the temperature of an enclosure and t the time. A and b are constants. This formula is called the Logarithmic Law of Cooling.

Prof. A. Weilenmann, of Zurich, applied Lambert's formula to the study of the variations of atmospheric temperatures at eight different places, and found that the value of b is almost constant for different localities, viz, (0.86—0.88). M. Alfred Angot,[8] of Paris, computed the value of b for all months of the year from the observational data of the nocturnal temperature at Paris. He found the mean value of $b=0.87$ for clear days and $b=0.86$ for cloudy days. Angot adopted 0.869 as the general mean of the values of b, and transformed Lambert's formula into the following form:

$$\theta=\theta_i+A\times 0.869^t \tag{2}$$

in which $t=0$ is for sunset.

$$\theta=\theta_i+A\times 0.869^t=\theta_i+A(e^{-.143})^t$$
$$=\theta_i+A\,e^{-.143t}.$$

For $t=\infty$, $e^{-.143t}=0$ and $\theta=\theta_i$, and for $t=0$, $A=\theta_0-\theta_i$ where θ_0 is its initial temperature. Therefore, Lambert's formula takes the form,

$$\theta=\theta_i+(\theta_0-\theta_i)\,e^{-.143t}. \tag{3}$$

Differentiating this formula, we have for the rate of nocturnal cooling,

$$\frac{\partial\theta}{\partial t}=Ab^t\log b=-.1405\times A\times 0.869^t$$

or

$$\frac{\partial\theta}{\partial t}=-.143\;(\theta_0-\theta_i)\;e^{-.143t}. \tag{4}$$

In 1872 Weilenmann introduced in his memoir,[9] "Ueber den täglichen Gang der Temperatur in Bern," the following differential equations:

$$\frac{\partial\theta}{\partial t}=-h(\theta-\theta'_{x=0}) \tag{5}$$

$$\left[\frac{\partial\theta'}{\partial t}\right]_{x=0}=-h(\theta'_{x=0}-\theta_i)+h(\theta-\theta'_{x=0}) \tag{6}$$

where θ is the temperature of the air, $\theta'_{x=0}$ that of the earth's surface, θ_i that of an imaginary upper stratum toward which the radiation of the earth's surface is supposed to take place, and h the emissivity of the earth's surface. Solving (5) and (6) simultaneously, Weilenmann found that, taking θ_i as constant,

$$\theta=\theta_i+c_1e^{-0.382ht}+c_2e^{-2.618ht}. \tag{7}$$

Then he computed the value of c_2 by applying the above formula to his observational data, and found that it is a very small number. Therefore the formula (7) reduces to the final equation,

$$\theta=\theta_i+c_1e^{-0.382ht} \tag{8}$$

which is identical with Lambert's formula, for Weilenmann

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[3] Six's Posthumous Works. Canterbury. 1794.

[4] Ann. de Chimie et de Physique. Third series. Vol. V.

[5] Pouillet, Éléments de Physique. Fourth edition, p. 610. 1844.

[6] Annales de Chimie et de Physique, 1848.

[7] For the sake of convenience, I shall use hereafter the following uniform notations:

	For atmosphere.	For earth.
Temperature	θ	θ'
Temperature of lowest stratum	$\theta_{x=0}$	$\theta'_{x=0}$ . . temperature of earth's surface.
Initial temperature	θ_0	θ'_0
Emissivity or coefficient of radiation	σ	σ'
Conductivity	k	k'
Diffusivity	$\frac{k}{\rho c}=a^2$	$a'^2=\frac{k'}{\rho' c'}$
Density	ρ	ρ'
Specific heat	c	c'
	$\frac{\sigma}{\rho c}=b^2$	$b'^2=\frac{\sigma'}{\rho' c'}$

ε = height of the stratum of invariable temperature in the air,
ε' = depth of the stratum of invariable temperature in the earth.
θ_i = temperature of imaginary athermanous stratum = sidereal temperature + upper atmospheric temperature.
θ_0 = initial temperature.
θ_m = mean temperature.
θ_{max} = maximum temperature.
θ_{min} = minimum temperature.
u = excess of temperature.
= time.

[8] "Influence de la Nébulosité sur la Variation diurne de la Temperature de Paris." Annales du Bureau Central, 1888.

[9] Schweizerische Meteorologische Beobachtungen, Bd. IX, 1872.

adopted .375 for the value of h. In his memoir he has finally shown how very closely his formula represents the natural phenomena of the nocturnal cooling of the atmosphere at Bern.

In the very beginning of his analysis, however, Weilenmann introduced a serious error by using the same coefficient h throughout in (5) and (6). In (5) h is evidently identical with $\frac{\sigma}{\rho c}$ in our notation, where σ is the coefficient of radiation of the atmosphere, ρ the density of the air, and c its specific heat. In the formula (6), however, h means $\frac{\sigma'}{\rho' c'}$ where σ' is the emissivity of the earth's surface. Hence, we see at once that h in (5) and in (6) can not be the same.

The problem was next attacked by Samuel Haughton, the Irish clergyman and scientist. In 1881 he published in the Transactions of the Royal Irish Academy two noteworthy papers under one title, "New Researches on Sun-heat and Terrestrial Radiation and on Geological Climates". In the first paper he treated the problem of the solar insolation and atmospheric radiation, adopting Rosetti's law of cooling[10]. The second paper has a direct connection with our problem. Empirically he came to the conclusion that at a temperature not much above θ_1 radiation is proportional to $\theta-\theta_1$; but at a temperature very much higher than θ_i radiation increases faster than this difference, and may be best represented by a parabolic curve, having its axis vertical and its vertex at the origin, viz:

$$p\,\frac{\partial\theta}{\partial t} = (\theta-\theta_1)^n, \tag{9}$$

where $n = 2.04$ and $p = 4.25$ for the month of January, which values were obtained from the mean of the 35 years of observation. He also reached the conclusion that nocturnal cooling is well represented by Newton's law. In this paper, however, Haughton adopted for the value of θ_i the lowest atmospheric temperature θ_{min} just before sunrise, because he thought that radiation must cease at that time. It appears that he followed the following argument:

$$\frac{\partial\theta}{\partial t} = K(\theta-\theta_1), \tag{10}$$

where K is a constant.
When θ becomes a minimum,

$$\frac{\partial\theta}{\partial t} = 0,$$

whence

$$K(\theta_{min}-\theta_1) = 0,$$

or

$$\theta_{min} = \theta_i.$$

But the integral curve of (10) is logarithmic and has no maximum or minimum point. Therefore, there is no point at which

$$\theta_i = \theta_{min}.$$

In 1884, Prof. William Ferrel, the first and the greatest theoretical meteorologist, published a memoir[11] on "The Temperature of the Atmosphere and the Earth's Surface" and later introduced it into his "Recent Advances in Meteorology". In them he treated the problem of the nocturnal cooling of the earth's surface and of bodies near it. Applying Petit and Dulong's law of cooling to his analysis, Ferrel obtained for the nocturnal cooling of the earth's surface the following formula:

$$\mu^{\theta'_{x=0}} - \theta_a = \mu^{\theta_a-\theta_i} - \frac{1}{B\sigma'\mu^{\theta_a}}\cdot\frac{dQ}{dt} \tag{11}$$

where

$\theta'_{x=0}$ = temperature of the earth's surface.
θ_a = temperature of atmosphere at 4 feet above the earth's surface.
θ_i = mean sky temperature, viz, the temperature of the imaginary athermanous stratum.
μ = 1.0077 one of Petit and Dulong's constants.
σ' = emissivity of the earth's surface.
B = rate at which heat is radiated from a unit surface of maximum emissivity, or of lampblack.

The expression $\frac{\partial Q}{\partial t}$ is the rate at which a unit of earth's surface is losing heat. According to this formula, $\frac{\partial Q}{\partial t}$ at first is comparatively large, but it gradually becomes less as the upper strata of the earth cool, but it never entirely vanishes. Therefore $\theta_a-\theta_i$ is the theoretical limit, to which $\theta'_{x=0}-\theta_a$ approximates, but can never quite reach, as $\frac{\partial Q}{\partial t}$ becomes smaller. This applies to the nocturnal cooling of the earth's surface, but it is to be regretted that Ferrel never touched the problem of the nocturnal cooling of the atmosphere itself.

Perhaps the most valuable contribution to our subject was made by Dr. Jules Maurer, of Zurich. Some twelve years after his countryman, A. Weilenmann, published his paper, Maurer dealt with the problem in a more analytical manner, and published the two important papers bearing the titles "Ueber die theoretische Darstellung des Temperaturgangs während der Nachtstunden and die Grösse der von der Atmosphäre ausgestrahlten Wärmemenge"[12] and "Temperaturleitung und Strahlung der ruhenden Atmosphäre"[13]. The first paper offers us some valuable formulæ and data; in particular he was the first to give us an approximate value of the coefficient of radiation by a cubic centimeter of air, which amounts to $.007\times10^{-4}$ cals. per minute. This Maurer deduced from the ordinary meteorological observations of the temperature of the atmosphere at nighttime, as given by the ordinary screened thermometers. Maurer finds indications that this coefficient is larger in summer and smaller in winter.

My careful examination of Doctor Maurer's memoirs, however, has revealed to me that he committed a serious mistake in his attempt to deduce an analytical formula for nocturnal temperature. He took advantage of Weilenmann's first equation

$$\frac{\partial\theta}{\partial t} = -\frac{\sigma}{\rho c}(\theta-\theta'_{x=0})$$

with which he deduced the formula,

$$\theta = \theta_i + c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t} \tag{12}$$

where c_1 and c_2, α_1 and α_2 are constants; α_2 is a function of ω_1 which is the first root of

$$\frac{\varepsilon'}{k}\cdot\tan\omega + \omega = 0, \tag{13}$$

where ε' is a depth below the earth's surface at which diurnal heat waves are supposed to vanish.

But if we refer to Riemann-Weber's Partielle Differentialgleichungen, Bd. II, § 52, we can at once see that this transcendental equation was obtained by Maurer from Fourier's surface condition for secular cooling:

$$\left[k'\,\frac{\partial u'}{\partial x}\right]_{x=0} = \sigma' u'$$

[10] This law is expressed by the formula
$\frac{\partial\theta}{\partial t} = (\theta-\theta_i)(a\Theta^2-\beta)$. Θ = absolute temperature; a and β are constants, and β is very small. From this Haughton deduced
$\frac{\partial\theta}{\partial t} = -B(\theta+460)^3$, where θ and Θ are expressed in degrees Fahrenheit.

[11] Professional Papers of Signal Service No. XII.

[12] Annalen der Schweiz. Meteorologischen Centralanstalt, Band XXII. 1885.

[13] Meteorologische Zeitschrift, 1886.

or

$$\left[k' \frac{\partial \theta'}{\partial x} \right]_{x=0} = \sigma' \left(\theta'_{x=0} - \theta_{x=0} \right). \qquad (14)$$

This surface condition presupposes that $\theta_{x=0}$ the temperature of the lowest strata of the atmosphere is constant, and that all heat conducted from the interior of the ground upward is radiated immediately by the surface that is exposed to the air of constant temperature. Hence, Maurer's solution can not be true and can not accord with the actual conditions of the problem. In his second paper[14] Maurer introduced an erroneous integral of the general differential equation[15] for heat conduction and radiation,

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 u}{\partial x^2} - \frac{\sigma}{\rho c} u \qquad (15)$$

where $u = \theta - \theta_1$, and σ is the coefficient of radiation.

Another important paper[16] on the subject was published in 1892 by Dr. W. Trabert, formerly of Vienna and now professor in Innsbruck. The interesting feature of this paper is the extensive computation of the value of Weilenmann's b for 42 stations of different localities and different elevations, and he found the value of σ to be $.0077 \times 10^{-5}$, which is very near that of Maurer. Further, he has drawn the conclusion that the value of σ is independent of the temperature and density of the air, and that the radiation of a unit mass of air is simply proportional to the absolute temperature.

In 1900, Dr. K. Nakamura, Director of the Central Meteorological Observatory of Tokio, read to the International Congress of Meteorology in Paris a short paper entitled " Sur la March Diurne de la Temperature de l'Air". The chief feature of this paper was an attempt to represent diurnal temperature by a formula advanced by the author. His equation for the nocturnal cooling is—

$$\frac{\partial \theta}{\partial t} = -\frac{\sigma}{\rho c} \left(\theta - \theta_j - p \sin t - q \cos t \right). \qquad (16)$$

Preceiving that θ_i in Newton's formula is not constant, Professor Nakamura substituted for it in Newton's law the expression $\theta_j - p \sin t - q \cos t$, where θ_j, p, and q are constants. The expressions $\sin t$ and $\cos t$ are, however, open to criticism, since it is impossible to take the sine and cosine of a time. Therefore, t must be multiplied by such coefficients as will transform t into an angle.

It will be of some interest here to mention von Bezold's important memoir[17] on the heat exchange at the earth's surface and in the air. The ground accumulates solar heat in summer, and gradually loses the stored energy in winter, so that there is a heat exchange during the year. A similar process occurs during the day and nighttime. According to von Bezold the heat content of the soil per unit area down to the depth ε of uniform temperature is

$$\int_0^{\varepsilon'} c \left(\theta' - \theta_0' \right) dx$$

where θ_0' is the initial temperature and θ' the momentary temperature at the depth of x and c the heat capacity. ε' is the depth of the invariable stratum. The difference of the maximum and minimum values of the above integral is the amount of heat exchange. Dr. J. Schubert[18] extended von Bezold's idea still further. He applied the above principle to the heat exchange in the atmosphere and on the ocean. For the atmosphere the heat content is

$$\int_0^{\varepsilon} \rho \, c_p (\theta - \theta_0) \, dx$$

or introducing air pressure b

$$1.36 \, c_p \int_0^{760} (\theta - \theta_0) \, db.$$

To this is to be added

$$0.6 \times 1.36 \int_0^{760} (e - e_0) \, db$$

where ε is the height of the stratum of invariable temperature in the atmosphere and e is the specific humidity of air. Schubert deduced the annual heat exchange in sandy soils, atmosphere, sea water, etc., and considered the effect of the earth's surface and ocean upon the temperature of the atmosphere. It will be interesting to apply these formulæ, or improved ones, if such be possible, to the study of daily heat exchange in the earth's surface and atmosphere.

Finally, I shall here add the titles of some memoirs and articles that bear some relation to our subject.

Fourier, Théorie Analytique de la Chaleur.

Poisson, Théorie Mathématique de la Chaleur. 1835.

Poisson, Mémoire sur les Température de la Partie solide du globe, de l'Atmosphere, etc. 1837.

Edmond Becquerel et Henri Becquerel, Mémoire sur la Température de l'Air a la Surface du Soil, etc. 1879.

J. Glaisher, On the Radiation of Heat at Night, from the Earth, etc. 1847.

H. Hennessy, On the Distribution of Temperature in the Lower Region of the Earth's Surface. 1867.

H. Wild, Ueber die Bodentemperatur in St. Petersburg und Nukuss. 1878.

H. Wild, Die Differenzen der Bodentemperatur, etc. St. Petersburg, 1897.

C. C. Hutchins, Radiation of Atmospheric Air. Am. Jour. Sci., Vol. XLIII, 1892.

Cleveland Abbe, Atmospheric Radiation of Heat and its Importance in Meteorology. Am. Jour. Sci., Vol. XLIII, 1892.

Frank Very, Atmospheric Radiation. Weather Bureau Bulletin. 1900.

Paul Campan, Essai sur le Refroidissant de l'Air et sur les Lois du Rayonnement. 1902.

H. A. Watson, Convection of Heat. Cambridge Phil. Soc. Proc. April, 1904.

C. C. Hutchins and J. C. Pearson, Air Radiation. Monthly Weather Review. July, 1904.

II. A MATHEMATICAL THEORY OF THE NOCTURNAL COOLING OF THE ATMOSPHERE NEAR THE EARTH'S SURFACE.

Hermann von Helmholtz attempted to prove in his famous memoir[19] "Ueber Atmosphärische Bewegungen" that by means of continually effective forces, there are formed in the atmosphere surfaces of discontinuity; that is to say, atmospheric strata of different density and temperature can lie contiguous, one above another, with a well defined surface of discontinuity between them. Recent meteorologists appear to reach the same conclusion by their balloon observations and kite experiments. In my mathematical analysis, however, I assume that the calm atmosphere is composed of an infinite number of very thin layers parallel to the earth's surface, that each stratum possesses the same temperature throughout, and that heat conduction, and radiation take place in one dimension only, viz, vertically.

Then the general equation for the nocturnal cooling of the atmosphere may be written as follows:

$$\frac{\partial \theta}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 \theta}{\partial x^2} - \frac{\sigma}{\rho c} (\theta - \theta_1),$$

[14] Meteorologische Zeitschrift, 1886.

[15] This differential equation and Maurer's second paper will be discussed later.

[16] Der tägliche gang der Temperatur und des Sonnenscheine auf dem Sonnblickgipfel. Denkschriften der Wien Acad. Bd LIX.

[17] "Der Wärmeaustausch an der Erdoberfläche und in der Atmosphäre." Sitzb. der kgl. preuss. Acad. d. Wiss. Berlin, 1892.

[18] " Der Wärmeaustausch in festen Erdboden, in Gewässern u. in der Atmosphäre. 1904.

[19] Sitzb. der kgl. preuss. Akad. d. Wiss. 1888. Also translated by Abbe. Smithsonian Institution. 1891.

or $$\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2} - b^2(\theta - \theta_1). \tag{1}$$

In addition to this there are two boundary conditions:

(a) For $x = 0$, or at the earth's surface, θ is directly influenced by the temperature of the earth's surface; $\theta = f(\theta')$ or there may be formed some equation like Fourier's surface condition.

(b) For $x = \varepsilon$, or at the stratum of invariable temperature, say 30 metres (after Wild), $\frac{\partial\theta}{\partial t} = 0$.

In integrating the differential equation (1) there are at least two difficulties. The first is that we do not know what functional relation exists between $\theta_{x=0}$ and $\theta'_{x=0}$. The second difficulty lies in our ignorance of the temperature gradient in the air or the nature of the function $\theta = f(x)$, when $t = 0$.

Maurer and Helmholtz have shown us how extremely insignificant is the effect of heat conduction upon the temperature of the atmosphere. Assuming that the temperature of the lowest stratum of the atmosphere is of simple harmonic type, Maurer used a particular solution of the differential equation

$$\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2} \tag{2}$$

namely

$$\theta = Ae^{-px}\cos\left(\frac{2\pi t}{T} - px + q\right), \tag{3}$$

where T is the period of temperature variation and A its amplitude. Substituting (3) for θ in equation (2), we obtain

$$p = \sqrt{\frac{\pi}{a^2 T}}.$$

Hence the rate v at which heat is propagated from one stratum to another must be

$$v = \frac{2\pi}{pT} = 2a\sqrt{\frac{\pi}{T}}. \tag{4}$$

Then Maurer took twelve hours of nighttime as a unit and for a^2 adopted Stefan's value,[20] which is 0.18 for one second. Hence for the half day, we have

$$a^2 = 0.18\times 60\times 60\times 12\ (\text{gram.}\times\text{cm.}^{-1}\times\tfrac{1}{2}\text{ day}^{-1}).$$

Therefore the rate of heat conduction

$$v = 2\sqrt{\pi\times 0.18\times 60\times 60\times 12} = 3.1 \text{ meters in twelve hours.} \tag{5}$$

Hence Maurer concluded that at this distance above the earth's surface heat waves disappear if the amplitude of heat waves at the surface does not exceed 10° or 20° and that the influence of nocturnal cooling of the earth's surface upon the temperature of air that is not very near to the earth's surface is inappreciable, so far as heat conduction is concerned.

Let us now integrate the equation (2) for thermal conduction of the atmosphere in one dimension,

$$\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2} \tag{6}$$

under the two conditions

(a) .... $\theta = 0$, for $x = 0$, or the temperature of air on the earth's surface is 0°.

(b) .. $\frac{\partial\theta}{\partial x} = 0$, for $x = H$, or the vertical gradient of temperature is zero at upper boundary surface of the atmosphere.

A particular solution, which satisfies (a), is

$$\theta = Ae^{-\alpha^2 a^2 t}\sin(\alpha x).$$

In order that this solution may satisfy (b) also, we must have

$$\theta = Ae^{-a^2\frac{\pi^2}{4H^2}t}\sin\left(\frac{\pi}{2H}x\right).$$

whence for $t = 0$,

$$\theta_0 = A\sin\left(\frac{\pi}{2H}x\right).$$

In order that $e^{-a^2\frac{\pi^2}{4H^2}t}$ may be equal to $\frac{1}{2}$, we must have

$$\frac{a^2\pi^2}{4H^2}t = \log_e 2 = 0.6932,$$

whence $$t = \frac{0.6932\times 4H^2}{a^2\pi^2} = 0.2809\frac{H^2}{a^2}$$

Adopting Maxwell's value for a^2 and 8000 meters for H, we reach the result,

t = 36,000 years (approximately),

That is to say,

$$\theta = A\sin\left(\frac{\pi}{2H}x\right) \text{ for } t = 0. \tag{7}$$

$$\theta = \tfrac{1}{2}A\sin\left(\frac{\pi}{2H}x\right) \text{ for } t = 36{,}000 \text{ years.}$$

From this it follows that an interval of 36,000 years would be necessary in order by conduction to reduce, by one-half, the difference of temperature between the bottom and top of the atmosphere. This is the conclusion reached by Helmholtz. In the above analysis, however, we must remember that we assumed that the temperature of the atmospheric stratum adjacent to the earth's surface is always equal to 0°, the temperature gradient $\frac{\partial\theta}{\partial x} = 0$ at $x = H = 8000$ meters, and that the whole atmosphere is of uniform density, $a^2 = \frac{k}{\rho c}$ being taken as constant. Therefore, Helmholtz's conclusion may not apply strictly, but it shows that the effect of heat conduction upon the temperature of the atmosphere is extremely insignificant, except near the earth's surface. On the other hand there are indications that Maurer's estimate of the effect of heat conduction is too large. He supposed that the variation of the atmospheric temperature very near the earth's surface is of simple harmonic nature and also that during the nighttime heat is propagated only by conduction. Radiation was not taken into account. Therefore, if we take account of atmospheric radiation, heat waves can not reach as high as three meters, but must die away long before they reach that height.

Doctor Wells gives us some very interesting information[21] on the formation of ice in India on nights when the air does not fall to the freezing point. Square pits about two feet deep and 30 inches wide were formed and filled with straw to a depth of about eight inches or one foot. On this rows of small unglazed earthen pans were placed, about $1\frac{1}{4}$ inches deep, filled with boiled soft water. Evaporation and radiation, acting together, cooled the water below freezing, and ice was formed. The natives of Bengal manufacture ice in large quantities by the practise of this method,[22] even when the temperature of the air is 16° to 20° F. above the freezing point. Doctor Wells pointed out that as the formation was most successful on calm, clear nights, the effect was due chiefly to radiation. This clearly shows how slow and insignificant the process of heat

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[20] In his memoir (Wien Sitzb, vol. 34) Stefan obtained for the value of the diffusivity of the air

$$a^2 = \frac{k}{c_v\rho} = \frac{0.0000558}{0.1686\times .001293} = 0.256\ [\text{cm.}^2\times\text{sec.}^{-1}].$$

But $a^2 = \frac{k}{c_p\rho}$ and not $\frac{k}{c_v\rho}$, where c_p is the specific heat of the air at constant pressure, and c_v at constant volume.

Therefore $a^2 = \frac{k}{c_p\rho} = \frac{0.0000558}{0.23788\times 0.001293} = 0.1812\ [\text{gm.}\times\text{cm.}^{-1}\times\text{sec.}^{-1}].$

[21] Poyning and Thomson, Heat, p. 233.

[22] Nature, January 4, 1872; also Ferrel's Recent Advances in Meteorology, p. 171.

conduction is when compared with radiation. Consequently we have to deal only with radiation in our problem of the nocturnal cooling of the atmosphere, except when the atmospheric strata very near the earth's surface are considered.

At first thought it appears that, according to Kirchhoff's law, any radiation from a mass of air may be of such a quality as to be immediately absorbed by the adjacent air in passing through it, and therefore can not escape into space, except when radiated by the very outermost stratum of the atmosphere. But the radiation by gaseous matter is very different from radiation by a solid or liquid. The latter radiates its heat from its surface only, but in the case of a gas every particle of it radiates into space directly. The experiments on radiation of ice and the observations of nocturnal frosts show that most rays of even such low temperatures can pass through thick strata of clear atmosphere without material absorption [23].

It is evident that the earth's surface at nighttime undergoes changes of temperature by the three processes:

a. Increase of temperature by conduction from the interior of the earth upward, that is, by the amount of

$$k'\left[\frac{\partial \theta'}{\partial x}\right]_{x=0} dt.$$

b. Increase of temperature by absorption from the lower strata of the atmosphere, or of

$$h\,(\theta_{x=0} - \theta'_{x=0})\,dt.$$

c. Decrease of temperature by radiation toward the sky, amounting to

$$-\sigma'\,(\theta'_{x=0} - \theta_i)\,dt.$$

Whence the change of temperature

$$[d\,\theta']_{x=0} = \left[k'\frac{\partial \theta'}{\partial x}\right]_{x=0} dt + h\,(\theta_{x=0} - \theta'_{x=0})dt - \sigma'\,(\theta'_{x=0} - \theta_i)\,dt$$

or

$$\left[\frac{\partial \theta'}{\partial t}\right]_{x=0} = \left[k'\frac{\partial \theta'}{\partial x}\right]_{x=0} + h\,(\theta_{x=0} - \theta'_{x=0}) - \sigma'\,(\theta'_{x=0} - \theta_i). \quad (8)$$

As we are going to deal with the relation between the temperatures of the earth's surface and the atmosphere later on, $h\,(\theta_{x=0} - \theta'_{x=0})$ may be dropped off from the expression (8). Moreover, the emissivity of the air is very small compared to that of the earth's surface and $h\,(\theta_{x=0} - \theta'_{x=0})$ may be insignificant. Then we have

$$\left[\frac{\partial \theta'}{\partial t}\right]_{x=0} = \left[k'\frac{\partial \theta'}{\partial x}\right]_{x=0} - \sigma'\,(\theta'_{x=0} - \theta_i). \quad (9)$$

For the interior temperature of the earth,

$$\frac{\partial \theta'}{\partial t} = a'^2\frac{\partial^2 \theta'}{\partial x^2}. \quad (10)$$

These two equations determine the temperature of the earth's surface. Putting $u_{x=0} = \theta'_{x=0} - \theta_i$ and $u = \theta' - \theta_i$, equations (9) and (10) become

$$\left[\frac{\partial u}{\partial t}\right]_{x=0} = \left[k'\frac{\partial u}{\partial x}\right]_{x=0} - \sigma' u \text{ for } x = 0. \quad (11)$$

$$\frac{\partial u}{\partial t} = a'^2\frac{\partial^2 u}{\partial x^2}. \quad (12)$$

In these equations x has been counted positive downward from the earth's surface. By observations, we know that there is a certain depth, say $x = \varepsilon'$, where there is no effect of the daily change of temperature at the surface, and such a layer we call the stratum of invariable temperature. Now let us measure x from this stratum upward. Then the equations (11) and (12) become

$$\left[\frac{\partial u}{\partial t}\right]_{x=\epsilon'} = \left[k'\frac{\partial u}{\partial x}\right]_{x=\epsilon'} - \sigma' u_{x=\epsilon'} \text{ for } x = \varepsilon'. \quad (13)$$

$$\frac{\partial u}{\partial t} = a'^2\frac{\partial^2 u}{\partial x^2}. \quad (14)$$

[23] Helmholtz, Ueber atmosphärische Bewegungen, 1888; also Abbe's "Mechanics of the Earth's Atmosphere," p. 89.

In addition to these, there enters the boundary condition,

$$\frac{\partial u}{\partial t} = 0 \text{ for } x = 0. \quad (15)$$

Now the expression

$$e^{-a'^2\lambda^2 t}\sin \lambda x \quad (16)$$

is a particular solution of the differential equation (14) and satisfies the condition (15) also.

Substitute (16) for $u_{x=\epsilon'}$ in (13), and we have

$$-a'^2\lambda^2 \sin \lambda\varepsilon' = k'\lambda \cos \lambda\varepsilon' - \sigma' \sin \lambda\varepsilon'$$

$$(\sigma' - a'^2\lambda^2) \sin \lambda\varepsilon' = k'\lambda \cos \lambda\varepsilon'.$$

Let $\omega = \lambda\varepsilon'$ or $\lambda = \frac{\omega}{\varepsilon'}$, then

$$\tan \omega = \frac{k'\omega\varepsilon'}{\sigma'\varepsilon' - a'^2\omega^2}. \quad (17)$$

The transcendental equation (17) has an infinite number of unequal roots; let them be denoted by ω_1, ω_2, ω_3, etc.

Then we have, for the general solution which satisfies our conditions, the following expression:

$$u_{x=\epsilon'} = A_1 e^{-\left(\frac{\omega_1}{\epsilon'}\right)^2 a'^2 t}\sin \omega_1 + A_2 e^{-\left(\frac{\omega_2}{\epsilon'}\right)^2 a'^2 t}\sin \omega_2 + \ldots\ldots$$

$$\theta'_{x=\epsilon'} = \theta_i + A_1 e^{-\left(\frac{\omega_1}{\epsilon'}\right)^2 a'^2 t}\sin \omega_1 + A_2 e^{-\left(\frac{\omega_2}{\epsilon'}\right)^2 a'^2 t}\sin \omega_2$$

$$+ A_3 e^{-\left(\frac{\omega_3}{\epsilon'}\right) a'^2 t}\sin \omega_3 + \ldots\ldots \quad (18)$$

If we take the first term of (18) for the first approximation

$$\theta'_{x=\epsilon'} = \theta_i + A_1 e^{-\left(\frac{\omega_1}{\epsilon'}\right)^2 a'^2 t}\sin \omega_1.$$

Differentiate this with respect to t, after taking logarithms,

$$\log\,(\theta'_{x=\epsilon'} - \theta_i) = \log A + \log \sin \omega_1 - \left(\frac{\omega_1}{\varepsilon'}\right)a'^2 t$$

$$\frac{d\,\theta'_{x=\epsilon}}{(\theta'_{x=\epsilon'} - \theta_i)} = -\left(\frac{\omega_1}{\varepsilon'}\right)^2 a'^2\,dt$$

whence

$$\left[\frac{\partial \theta'}{\partial t}\right]_{x=\epsilon'} = -\left(\frac{\omega_1}{\varepsilon'}\right)^2 a'^2\,(\theta'_{x=\epsilon'} - \theta_i) \quad (19)$$

which is the first approximation to the formula for the nocturnal cooling of the earth's surface.

We may assume that the nocturnal cooling of the atmosphere near the earth's surface is caused principally by the cooling of the latter. Then we have the relation,

$$\frac{\partial \theta}{\partial t} = -b^2\,(\theta - \theta'_{x=\epsilon'}). \quad (20)$$

Now if we solve (19) and (20) simultaneously

$$\theta = \theta'_{x=\epsilon'} + pe^{-b^2 t}$$

$$\theta_{x=\epsilon'} = \theta_i + qe^{-\left(\frac{\omega_1}{\epsilon'}\right)^2 a'^2 t}$$

whence

$$\theta = \theta_i + pe^{-b^2 t} + qe^{-\left(\frac{\omega_1}{\epsilon'}\right)^2 a'^2 t}$$

$$\text{or} \qquad \theta = \theta_i + pe^{-\frac{\sigma}{\rho c}t} + qe^{-\left(\frac{\omega_1}{\epsilon'}\right)\frac{k'}{\rho' c'}t} \quad (21)$$

where ω_1 is the first root of

$$\tan \omega = \frac{k'\omega\varepsilon'}{\sigma'\varepsilon'^2 - a'^2\omega^2} \quad (22)$$

p and q may be easily determined.

Putting

$$\begin{cases} \beta = b^2 = \dfrac{\sigma}{\rho c} \\ \alpha_1 = \left(\dfrac{\omega_1}{\varepsilon'}\right)^2\dfrac{k'}{\rho' c'}, \end{cases}$$

(21) becomes

$$\theta = \theta_i + pe^{-\beta t} + qe^{-\alpha_1 t}. \quad (23)$$

If $t=0$, $\theta=\theta_0$

$$\therefore \quad \theta_0 = \theta_i + p + q$$

$$\theta_0 - \theta_i = p + q. \quad (24)$$

Differentiate (23) with respect to t,

$$\frac{\partial \theta}{\partial t} = -\beta p e^{-\beta t} - \alpha_1 q e^{-\alpha_1 t}.$$

If the initial rate of cooling is represented by r, the last equation becomes

$$\left[\frac{\partial \theta}{\partial t}\right]_{t=0} = r = \beta p + \alpha_1 q. \tag{25}$$

By solving (24) and (25) simultaneously, we obtain

$$p = \frac{\alpha_1(\theta_0 - \theta_i) + r}{\alpha_1 - \beta}$$

$$q = \frac{\beta(\theta_0 - \theta_i) + r}{\beta - \alpha_1}.$$

Hence our equation (23) becomes

$$\theta = \theta_i + \frac{\alpha_1(\theta_0 - \theta_i) + r}{\alpha_1 - \beta} e^{-\beta t} + \frac{\beta(\theta_0 - \theta_i) + r}{\beta - \alpha_1} e^{-\alpha_1 t} \tag{26}$$

or

$$\theta = \theta_i + \frac{\left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'}(\theta_0 - \theta_i) + r}{\left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'} - \frac{\sigma}{\rho c}} e^{-\frac{\sigma}{\rho c} t} + \frac{\frac{\sigma}{\rho c}(\theta_0 - \theta_i) + r}{\frac{\sigma}{\rho c} - \left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'}} e^{-\left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'} t}$$

and

$$\frac{\partial \theta}{\partial t} = -\left\{ \beta \frac{\alpha_1(\theta_0 - \theta_i) + r}{\alpha_1 - \beta} e^{-\beta t} + \alpha_1 \frac{\beta(\theta_0 - \theta_i)}{\beta - \alpha_1} e^{-\alpha_1 t} \right\}. \tag{27}$$

The equation (27) represents the rate of cooling of the atmosphere at night under our assumed conditions which closely approximate nature in the temperate and tropical zones. For $t = 0$, we get

$$\left[\frac{\partial \theta}{\partial t}\right]_{t=0} = r.$$

These equations take the same form as Maurer's equations, but my transcendental equation is $\tan\ \omega = \frac{k' \omega \varepsilon'}{\sigma' \varepsilon'^2 - a'^2 \omega^2}$ while Maurer's is $\tan\ \omega = \frac{k' \omega}{\sigma' \varepsilon'}$ which was deduced under the erroneous assumption pointed out in my first paper.

If we take more than one root of the transcendental equation, we may be able to express θ in the following series:

$$\begin{aligned} \theta_0 = \theta_i &+ \frac{\alpha_1(\theta_0 - \theta_i) + r}{\alpha_1 - \beta} e^{-\beta t} + \frac{\beta(\theta_0 - \theta_i) + r}{\beta - \alpha_1} e^{-\alpha_1 t} \\ &+ \frac{\alpha_2(\theta_0 - \theta_i) + r}{\alpha_2 - \beta} e^{-\beta t} + \frac{\beta(\theta_0 - \theta_i) + r}{\beta - \alpha_2} e^{-\alpha_2 t} \\ &+ \text{etc.} \end{aligned} \tag{28}$$

where as before

$$\beta = \frac{\sigma}{\rho c}$$

$$\alpha_1 = \left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'}$$

$$\alpha_2 = \left(\frac{\omega_2}{\varepsilon'}\right)^2 \frac{k'}{\rho' c'}$$

$$\alpha_3 = \left(\frac{\omega_3}{\varepsilon'}\right) \frac{k'}{\rho' c'} \text{ etc.}$$

These equations (26), (27), (28) represent the nocturnal variation of the temperature and the rate of the nocturnal cooling of the atmosphere near the earth's surface. It is apparent from these equations that both the temperature variation and the rate of the cooling depend upon the coefficients, $a'^2 = \frac{k'}{\rho' c'}$, the diffusivity of the earth, σ', its emissivity, σ, the radiating power of the atmosphere, ρ, its density, and c, its specific heat, and also ε', the depth of the stratum of invariable temperature in the earth. These coefficients may vary with the change of temperature, but, as the nocturnal variation of temperature is comparatively small, they may be supposed constant, which I believe is approximately correct.

Nocturnal cooling and temperature variation also depend upon θ_i and r.

Approximate values of physical constants:

$a'^2 = \frac{k'}{\rho' c'} = 0.50$ gr. cm./min. = 30.00 gr. cm./hr., for earth [Wild].
$k' = 0.30$ gm. cm./min. = 18.00 gr. cm./hour, for earth [Wild].
$\sigma' = 0.006$ gm. cm./min. = 0.36 gr. cm./hour, for earth [Stefan].
$\varepsilon' = 1$ meter. [Wild].
$b^2 = \frac{\sigma}{\rho c} = 0.13$ gr. cm./hour, for air. [Maurer].

There is much to be said about these physical constants. None of these values, given above or elsewhere, can be trusted absolutely. They await more accurate redeterminations by physicists and meteorologists. The most doubtful constant of all is σ, the coefficient of radiation by the air. According to Maurer, a cubic centimeter of air at 0° and 760$^{\text{mm}}$ of pressure radiates $.007 \times 10^{-4}$ cals. per minute, but according to Trabert, $.0077 \times 10^{-5}$ cals. Thus the radiation per hour of one gram of air at 0° C. toward a surface of absolute 0° temperature is 9.83 cals. Maurer finds indications that his coefficient is larger in summer and smaller in winter and that the coefficient found for the air saturated with moisture, or mixed with dust, is larger than belongs to a pure, dry air. He further notes that similar computations based on observations made at high stations, St. Bernard and Santis, give a coefficient about 15 per cent smaller. On the other hand, Trabert draws the conclusion from his observations that the radiation of a unit mass of air is simply proportional to the absolute temperature and that the coefficient of atmospheric radiation is independent of the density of air. By reason of all these observations and computations, there arises a doubt whether the values of σ, obtained by Maurer and Trabert, represent the actual radiation coefficient. Some years ago, Prof. C. C. Hutchins, of Bowdoin College, attempted to determine this coefficient by experimental methods. His value is 100 times larger than those of Maurer and Trabert. It is evident that Hutchins's value is too large, and Maurer's and Trabert's values are too small. Maurer's computation depends upon the Newtonian law and considers average radiation during cloudy, as well as clear, nights, while the metallic tube in Hutchins's experiments was overheated and may have given off an extra quantity of dust or of gaseous compounds, all of which tend to increase the radiation. Hutchins's latest measures are given in the American Journal of Science (4), vol. 18, October, 1904, pp. 277–286. C. C. Hutchins and J. C. Pearson, "Air Radiation." This memoir is reprinted in the MONTHLY WEATHER REVIEW for July, 1904.

According to Wild,[24] the diffusivity a'^2 of the earth, like black earth, sand, clay, etc., is of constant value 0.50 grm. cm./min., at the average temperature 11° C, and the conductivity, k', may be accepted as 0.3 approximately.

If we assume that the earth's surface has a maximum emissivity, we may be able to calculate its value from Stefan's result of experiments on the emissivity of a black body[25]. According to Stefan, the difference of the quantities of heat emitted by a black surface of 1 sq. cm. at 100° C and at 0° C is approximately 1 calorie, or

$$A\,[\,(273+100)\,]^4 - (273)^4\,] = 1 \text{ cal.}$$

whence

$$A = .7245\ldots \times 10^{-10} = .725 \times 10^{-10}$$

$$\sigma' = .725 \times 10^{-10} \times [\,(273+1)^4 - 273^4]$$

$$= .006 \text{ cal. per minute.}$$

The depth of the invariable stratum ε' is 1 m., according to Wild, and at this depth the change is only one hundredth of one degree centigrade during the day.

[24] "Ueber die Boden temperaturen in St. Petersburg und Nukuss." 1878–79.

[25] "Ueber die Beziehung zwischen der Wärmestrahlung u. die. Temperatur." Sitzb. Kgl. Akad. Wiss. 1879.

Taking these constants as true, our transcendental equation becomes

$$\tan \omega = \frac{\kappa_1\, \omega\, \varepsilon'}{\sigma' \varepsilon'^2 - a_1^2 \omega^2} = \frac{18 \times 100\, \omega}{(0.36 \times 100^2 - 30\, \omega^2)}$$

$$= \frac{60\, \omega}{120 - \omega^2}.$$

Now in order to find the roots of this equation graphically we put

$$y = \tan \omega \qquad (a)$$

and

$$y = \frac{60\, \omega}{120 - \omega^2} \qquad (b)$$

and construct two sets of curves taking y as ordinates and ω as abscissas. Then the values of the abscissas of the intersecting points of curves will be the roots of $\tan \omega = \frac{60\, \omega}{120 - \omega^2}$.

The equation (a) is a well known transcendental equation. The second one presents a new curve, whose characteristic points are

$$y = 0 \text{ for } \omega = 0.$$
$$y = 0 \text{ for } \omega = \infty.$$
$$y = \infty \text{ for } \omega^2 = 120 \text{ or } \omega = 10.95.$$

For maximum or minimum points,

$$\frac{dy}{d\omega} = 60\, \frac{120 + \omega^2}{120 - \omega^2} = 0$$

whence

$$120 + \omega^2 = 0$$
$$\omega = \sqrt{-120}.$$

Hence, we see that there is no maximum or minimum. For $y = \tan \omega$ we compute the following table:

ω	*In radians.*	*In angle.*	*Tan* ω.
$\frac{1}{10}\left(\frac{\pi}{2}\right)$	= .1571	9°	.1584
$\frac{2}{10}\left(\frac{\pi}{2}\right)$	= .3141	18°	.3249
$\frac{3}{10}\left(\frac{\pi}{2}\right)$	= .4712	27°	.5095
$\frac{4}{10}\left(\frac{\pi}{2}\right)$	= .6282	36°	.7265
$\frac{5}{10}\left(\frac{\pi}{2}\right)$	= .7854	45°	1.0000
$\frac{6}{10}\left(\frac{\pi}{2}\right)$	= .9424	54°	1.3764
$\frac{7}{10}\left(\frac{\pi}{2}\right)$	= 1.0995	63°	1.9626
$\frac{8}{10}\left(\frac{\pi}{2}\right)$	= 1.2564	72°	3.0777
$\frac{9}{10}\left(\frac{\pi}{2}\right)$	= 1.4139	81°	6.3138
$\frac{\pi}{2}$	= 1.5708	90°	∞

For $y = \frac{60\omega}{120 - \omega^2}$ we compute the following values:

ω	y	ω	y
1	0.50	7	6.00
2	1.03	8	8.55
3	1.62	9	13.80
4	2.30	10	30.00
5	3.16	10.95	∞
6	4.30	12	−30.00

ω	y	ω	y
14	−11.05	28	−2.50
16	− 7.60	30	−2.30
18	− 5.30	40	−1.60
20	− 4.30	50	−1.26
.22	− 3.62	60	−1.03
24	− 3.15	100	−0.60
26	− 2.80	200	−0.30

Plotting these values, will give two sets of curves like those in the accompanying diagram, fig. 1.

$$\frac{177}{128}\pi = 4.344 \text{ for } \omega_1$$

$$\frac{313}{128}\pi = 7.685 \quad \text{"} \quad \omega_2$$

$$\frac{7}{2}\pi = 10.995 \quad \text{"} \quad \omega_3$$

etc., etc.

$$\beta = .1300$$

$$a_1 = \left(\frac{\omega_1}{\varepsilon'}\right)^2 \frac{k_1}{\rho' c'} = \left(\frac{4.344}{100}\right)^2 \times 30 = 0.0568.$$

Hence,

$$\theta = \theta_i - \frac{.0568\,(\theta_0 - \theta_i) + r}{.0732}\, e^{-.1300\,t} + \frac{.1300\,(\theta_0 - \theta_i) + r}{.0732}\, e^{-.0568\,t}$$

$$\omega_2 = 7.685.$$

$$a_2 = \left(\frac{7.685}{100}\right)^2 \times 30 = .1776.$$

$$\left\{\begin{aligned} &\frac{a_2(\theta_0 - \theta_i) + r}{a_2 - \beta}\, e^{-\beta t} = \frac{.1776(\theta_0 - \theta_i) + r}{.0476}\, e^{-.1300\,t} \\ &\frac{\beta(\theta_0 - \theta_i) + r}{\beta - a_2}\, e^{-a_2 t} = -\frac{.1300(\theta_0 - \theta_i) + r}{.0476}\, e^{-.1776\,t} \end{aligned}\right.$$

$$\omega_3 = \left(\frac{10.995}{100}\right)^2 \times 30 = 0.3628$$

$$\left\{\begin{aligned} &\frac{a_3(\theta_0 - \theta_i) + r}{a_3 - \beta}\, e^{-\beta t} = \frac{.3628(\theta_0 - \theta_i) + r}{.2328}\, e^{-.1300\,t} \\ &\frac{\beta(\theta_0 - \theta_i) + r}{\beta - a_3}\, e^{-a_3 t} = -\frac{.13(\theta_0 - \theta_i) + r}{.2328}\, e^{-.3628\,t} \end{aligned}\right.$$

Summing up these terms we obtain,

$$\theta = \theta_i + \left\{ -\frac{.0568(\theta_0 - \theta_i) + r}{.0732} + \frac{.1776(\theta_0 - \theta_i) + r}{.0476} + \frac{.3628(\theta_0 - \theta_i) + r}{0.2328} \right\} \times e^{-.13t} + \left(.13\,(\theta_0 - \theta_i) + r\right) \times$$
$$\times \left\{ \frac{1}{.0732}\, e^{-.0568\,t} - \frac{1}{.0476}\, e^{-.1776\,t} - \frac{1}{.2328}\, e^{-.3628\,t} \right\}$$

The meaning of θ_i is already explained elsewhere, but as regards its nature and magnitude, θ_i gives rise to a multitude of questions. Let us imagine for a moment that an elementary volume of air is exposed in the atmosphere to nocturnal radiation. Then this volume will receive heat from two different sources, namely, from interplanetary space and from the atmosphere, if the radiation from the earth's surface is cut off. The sidereal heat, which name we owe to Poisson, is the total heat which reaches the earth from all celestial bodies excepting the sun. This heat is called star heat by Haughton; temperature of space, or weltraumstemperatur, by Frölich. Fourier was the first to show that it is necessary to take account of this sidereal heat in order to explain the phenomena of nocturnal radiation, and to indicate in a general way that the heat ought to be very little inferior to the temperatures of the poles of the earth, and about 50° or 60° below zero C. Poisson believed that this heat is higher than that pointed out by Fourier and concluded that it differs very little from zero temperature and that it has the same intensity in all directions, when it reaches the earth. On the other hand, Pouillet and Frölich attempted to separate the constant effect of the sidereal heat from the variable effect of the atmosphere; and Pouillet found the temperature of sidereal heat to be —142° C., and Frölich —131° C. from his St. Petersburg observations. With respect to the heat emitted by the atmosphere itself in the course of the night, it is the effect of the individual radiations of the concentric upper strata of the atmosphere. It depends, consequently, upon the distribution of temperature in upper atmospheric regions.

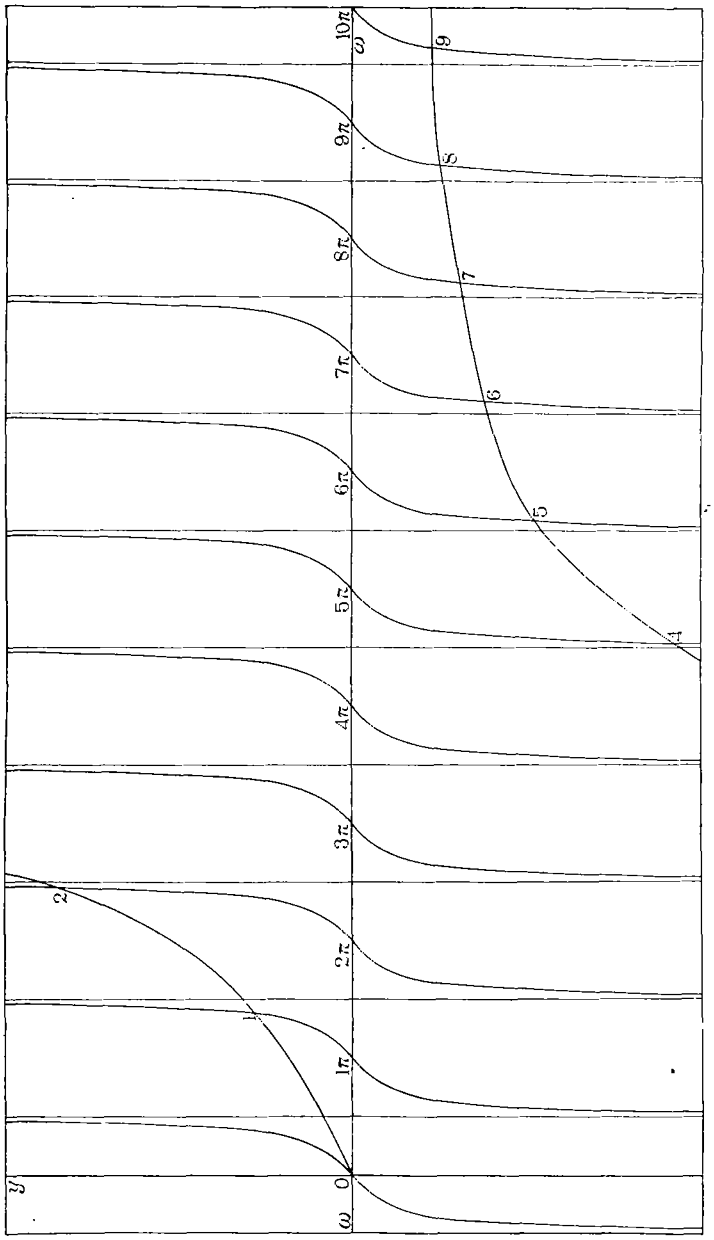

FIG. 1.—Graphic solution of the transcendental equation.

Whatever be the relation of the intensities of the two causes just mentioned, it is evident that we may conceive of a single cause producing an effect equal to that which results from the simultaneous action, or, in other words, we may suppress the sidereal heat and the heat of the upper atmosphere, and conceive an *imaginary athermanous surface*, of maximum emissive power, the temperature of which is such that it imparts to the elementary volume of air as much as it receives from the upper atmosphere and from space. This unknown temperature was called the zenithal temperature by Pouillet, the sky temperature by Ferrel, and himmelstemperatur by Frölich. I prefer to call it the temperature of the imaginary athermanous surface, and it is denoted by the notation θ_i throughout the present analysis. According to Haughton, the mean value of θ_i is —108.16° F. Frölich, who has recently made important researches on the subject, with greatly improved instruments, at St. Petersburg, has obtained about the same value as that of Haughton. It is evident that since the sidereal heat is constant, but the heat emitted by the atmosphere is variable, the temperature of the imaginary athermanous surface must experience some variation. Pouillet's experiments with his actinometer during very beautiful, clear nights show that the temperature θ_i is lowered during the night, from the setting to the rising of the sun. This variation of θ_i, however, can not be very great, as the temperature of the atmosphere by the upper regions is constant through all the seasons and all the night. According to Teisserenc de Bort's observations in his 581 balloon ascensions, the temperature gradient of the atmosphere becomes smaller the higher we ascend, and at last the temperature becomes constant throughout all the seasons at the height of 8000 meters. Professor Wild observed that the daily oscillation of the temperature becomes smaller as we go upward, and vanishes at the height of 30 meters. Hence it is clear that that part of the atmosphere which may be the cause of the variation of θ_i during the night is only a thin stratum. Moreover, the emissive power of the atmosphere is a very small quantity, and so the variation of θ_i in clear weather, if it exist at all, must be very small. It is, however, highly important to determine the exact value of θ_i experimentally or otherwise, and to investigate its variation and also the ratio or any other relation of the sidereal heat to the heat emitted by the upper atmosphere.

If we know the physical constants for ice, we can apply the formulæ developed above to the study of the nocturnal cooling of the atmosphere over ice surfaces, by replacing the coefficients for the earth by those for ice. Or, if the earth's surface is covered by a very thin layer of snow, we can deal with the problem by replacing only the emissivity for the earth's surface by that for snow.

In conclusion, a few words must be said. In the first paper, I have given a historical summary of observational work on the nocturnal cooling of the atmosphere by Wilson, Six, Wells, Pouillet, Melloni, and others; and then pointed out the errors and weak points in the theories of the problem advanced by Lambert, Weilenmann, Haughton, Maurer, and Nakamura. In the second paper, I have, by quoting from Maurer's, Helmholtz's, and Wells's work, shown how extremely small is the effect of heat conduction upon the nocturnal cooling of the atmosphere, and have also shown that Maurer's estimation of the effect is too large and Helmholtz's is too small. Neglecting the effect of heat conduction, I have now attempted to solve the problem by taking advantage of Weilenmann's and Maurer's differential equations, but have adopted a more legitimate assumption as regards the thermal relation between the atmosphere and the earth's surface, since Maurer's assumption is erroneous. I have thus obtained a new transcendental equation, whose roots furnish new values for the solution of the differential equations. Then I have discussed the physical constants which come in the analysis and reduced my general formulæ into numerical expressions by introducing the approximate values of the physical constants. Finally, the meaning of θ_i, the imaginary athermanous surface, was explained and the importance of further research on its magnitude and variation was pointed out.

As already explained elsewhere, the problem of the nocturnal cooling of the atmosphere is extremely complex and difficult for modern mathematical analysis. It appears to me that the phenomena of the atmospheric radiation may not yield to analytical methods so long as the necessary physical bases are lacking. Neither of the coefficients adopted above is absolutely reliable. Besides, in order to complete the solution of the problem and verify the results of mathematical analysis, we must have at hand the results of simultaneous nocturnal observations of the temperature of the atmosphere at different heights (say, from $x = 0$ to $x = 30$ meters) and of the temperature of the earth's surface and the soil at different depths. We need accurate knowledge of the position of the strata of invariable temperature in the earth and in the atmosphere. As pointed out elsewhere, we must also know more completely the magnitude and variation of the imaginary athermanous surface and of its functional relation to the sidereal heat and the temperature of the upper atmosphere. This paper does not pretend to contribute anything toward the final solution of the problem; but in the present status of our knowledge of the subject, even an approximate analysis may yet prove of importance. This statement is justified by Lord Rayleigh's introductory words in his memoir on the Vibrations of an Atmosphere:[26]

> In order to introduce greater precision into our ideas respecting the behavior of the earth's atmosphere, it seems advisable to solve any of the problems that present themselves, even though the search for simplicity may lead us to stray rather far from the actual question.

In preparing the account of this research, I am indebted to many writers, from whose books and memoirs I freely quoted in order to support my views on the subject. I would especially mention Dr. J. Maurer's important memoirs, often mentioned previously. I take the advantage of this opportunity to express my gratitude for valuable criticisms and suggestions, to Prof. R. S. Woodward, now President of the Carnegie Institution, whose masterly command of mathematical analysis in physical inquiry and whose clear and lucid exposition have been the source of inspiration that led me to pursue the study of mathematical physics with great enthusiasm; and to the Editor of the MONTHLY WEATHER REVIEW, under whose kind direction I have been engaged in meteorological research and who proposed the present problem[27] for investigation. I have also to thank Prof. A. Graham Bell, of Washington, D. C., and Dr. L. A. Bauer, the Director of the Department of Terrestrial Magnetism of the Carnegie Institution, for their cordial assistance in many ways.

THE INFLUENCE OF SMALL LAKES ON LOCAL TEMPERATURE CONDITIONS.

By JAMES L. BARTLETT, Observer Weather Bureau. Dated April 25, 1905.

The city of Madison, Wis., is situated between Lakes Mendota and Monona on a strip of land trending northeast and southwest and varying in width from one-half to three-fourths of a mile. Since April, 1883, meteorological observations have been taken at Washburn Observatory, which is located at one end of this strip on a slight ridge overlooking Lake Mendota, the larger of the two lakes; the observatory is 100 feet above and 600 feet distant from this lake. Besides the two lakes mentioned there are several smaller ones in the vicinity, so that within a radius of five miles from the observatory the surface is about one-third water. It would therefore appear to be a very favorable location for observing any appreciable effects that the lakes may have upon the local air temperature.

To obtain an accurate knowledge of these effects it was found necessary to compare the Washburn Observatory temperatures with those of neighboring points which have no lakes in their vicinity. For this purpose the records at the four cooperative observation stations nearest to Madison were selected. These are Harvey, Portage, Beloit, and Dodgeville, located, respectively, to the east, north, south, and west of Madison and within a radius of 45 miles from the last-named city.

For periods of the same lengths and dates in each case for Madison and for the point under consideration the following temperature data for each calendar month were computed: mean, mean maximum, mean minimum, and mean daily range. Corrections for the differences in the mean annual temperatures of the various places were then applied. Thus, the mean annual temperature at Beloit was found to be 1.3° higher than that at Madison for the period under consideration; this amount was therefore deducted from all of the Beloit data, except the mean daily range. The resulting values were presumed to show the temperature conditions which would exist at Madison were there no lakes in its vicinity. The departures of the Madison temperature data from the corrected data of the other points were then computed and are plotted below. The departures are shown in degrees Fahrenheit and are positive above the zero line, negative below. The curves, reading downward in each case, show the departures, respectively, from the Harvey, Portage, Beloit, and Dodgeville data.

The departure curves of fig. 1 show certain general points of agreement. The mean maximum and mean daily range curves have a negative departure, and the mean minimum a positive departure during nearly all the year in each case. The mean monthly curve has a general negative departure during the first five months and a general positive departure the remainder of the year. The mean minimum reaches its extreme positive departure in August in each case. It is believed that the irregularities of certain of the curves, particularly of those of the mean maximum, are due to local peculiarities of the climate of the various points. Thus, Beloit and Harvey are nearer to, and therefore their temperatures are more under the influence of, Lake Michigan than is that of Madison. Dodgeville is more remote from the great lake and less affected by it. Portage is in the valley of the Wisconsin River, which doubtless has some influence upon local temperature conditions. To eliminate these irregularities as far as possible an average of the departures was plotted for each element of the temperature data, and this may now be considered.

The monthly mean departure curve (fig. 1, No. 19) shows quite clearly the slight influence which the lakes have in retarding the annual increase of temperature in the spring and its decrease in the fall. This seems to be manifested in the spring by a lowering of the maximum or day temperatures, and in the fall chiefly by a raising of the minimum or night temperatures.

The influence of the lakes in preventing the occurrence of killing frosts late in the spring and early in the fall is quite marked, and is indicated by the positive departure of the minimum curve from April to October. The average date of the last spring killing frost at Madison is April 21; the average date for Harvey, which has nearly the same latitude as Madison and is nearer Lake Michigan, during the past thirteen years has been over two weeks later. In the fall killing frosts occur in general two weeks earlier at Harvey than at Madison. It would thus appear that the growing season is considerably lengthened at Madison by the presence of the lakes.

During January and February the lakes are almost invariably thickly coated with ice, and presumably the local exposure then becomes purely continental. That the frozen lakes do exert some control upon the temperature of the overlying air is shown by the fact that the range of temperature at Madison during these months still averages over two degrees less than at points away from the lakes. Also the monthly mean is slightly below the assumed purely continental type at this time, but it is possible that this should not be the case. It seems more reasonable to believe that the whole mean

[26] Phil. Mag., Feb., 1890. Abbe's Mechanics of the Atmosphere, p. 289.

[27] The third and fourth papers on the subject will be published in a subsequent number of the Monthly Weather Review.